

# Free Massless Particles, Two Time Physics and Newtonian Gravitodynamics

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## Abstract

We demonstrate how a classical Snyder-like phase space can be constructed in the Hamiltonian formalism for the free massless relativistic particle, for the two-time physics model and for the relativistic Newtonian gravitodynamic theory. In all these theories the Snyder-like phase space emerges as a consequence of a new local scale invariance of the Hamiltonian. The implications and consequences of this Snyder-like phase space in each of these theories are also considered.

## 1 Introduction

Topological and geometrical effects are among the most striking quantum phenomena discovered since the formulation of quantum mechanics in 1926. The first of such effects was discovered in 1959 and is termed the Aharonov-Bohm effect [1]. This effect caused a revolution in the basic assumptions about the role of potentials in physics. Its discovery prefigured a revolution in our understanding of gauge fields and of the fundamental forces of nature [2].

In classical electrodynamics the scalar and vector potentials are no more than mathematical conveniences, mere auxiliary fields that we can take or leave. Only the electric and magnetic fields, which act locally on the charges, are physical fields. In quantum physics the potentials are much more than mathematical conveniences. It is clear that whenever scalar and vector potentials appear in the classical Hamiltonian for a physical system, they will also appear in the corresponding Schrodinger equation, in the Heisenberg equations and in the Feynman path integral [2]. One of the results we present in this work is the verification that, in a flat Euclidean or in a noncommutative space-time, we can have non-vanishing classical brackets among the canonical variables that describe the motion of a massless relativistic particle and the components of a relativistic vector potential  $A_\mu$  that describes a Newtonian gravitodynamic field.

In the corresponding quantized theory, with these brackets turned into commutators, there will exist non-vanishing uncertainty relations between the massless particle's canonical variables and the components of the gravitodynamic potential. These uncertainty relations are the evidences of the physical nature of the gravitodynamic potential in quantum mechanics. The brackets we present in this work are then the evidences of the physical nature of the gravitodynamic potential in classical physics. Before turning to these subjects, let us explain the motivation for this work.

There is at the moment a considerable amount of information theorists expect that a consistent quantum theory of gravity should incorporate. To begin with, such a theory should incorporate the consequences [3] of the holographic principle. This principle requires that the density of the available information about quantum gravity in a certain space-time region be bounded by the surface surrounding that region. The density of information on the surface can not exceed one bit of information per Planck area.

There is now reasonable confidence that quantum gravity should be formulated in a noncommutative space-time. This comes in as a consequence of additional quantum uncertainties among the canonical variables that are introduced by gravitational effects at the Planck length scale [4]. The first attempt to define a Lorentz invariant noncommutative space-time was performed by Snyder [5] in 1947. Snyder's noncommutative quantized space-time was originally proposed as way to solve the ultraviolet divergence problem in quantum field theory and has recently attracted some interest of researchers working in quantum gravity and in the physics of mesoscopic systems.

A Snyder-like space-time for quantum gravity can bring new light in the context of the connection between the infrared divergences of a theory containing gravity in a certain region of space-time and the ultraviolet divergences in the conformal field theory living on the boundary of this region, the so called IR/UV connection, or anti-de Sitter/conformal field theory (AdS/CFT) connection, which is one of the cornerstones [6] of the holographic principle. Some time ago, a new de Sitter/Snyder-Yang space-time connection was proposed [7] in substitution to the AdS/CFT correspondence. The idea behind this proposal is that conformal field theory entirely lacks the space-time noncommutativity necessary to make quantum gravity a finite theory. Snyder-like brackets in  $D$  dimensions were recently derived [8] in the reduced phase space of the  $D+2$  dimensional two-time physics model [9,10,11,12] using the Dirac bracket technique [15] for the quantization of systems with second class Hamiltonian constraints. This result is encouraging because the two-time physics model is classically equivalent to (0+1) dimensional conformal gravity [13,14].

Despite the harmony between all the physical ideas about quantum gravity described above, a basic difficulty remains unresolved: which physical object is to be considered as the fundamental bit of information occupying the Planck area? In this work we support the idea that a relativistic gravitational dipole at the Planck length can be turned into a possible candidate for such a fundamental object. As pointed out in [16], the concept of a free massless relativistic point particle becomes ambiguous in a noncommutative space-time. Our proposal

is that a way out of this difficulty is to admit that a free massless relativistic particle in a noncommutative space-time becomes dynamically equivalent to an extended relativistic object whose spatial extension is a measure of the length scale at which space-time becomes non-commutative. The simplest possible extended relativistic object with an effective zero mass is a gravitational dipole, composed of two very small opposite masses separated by a very small distance. Negative masses [17] can mathematically appear in massless relativistic particle theory as a consequence of a non-vanishing minimal space-time length. A brief and rough discussion of the model of a free massless relativistic particle as a gravitational dipole at the Planck length can be found in [18].

According to Newtonian gravitation opposite masses should repel each other and so the dipole structure is unstable, but it can become stable if other effects are introduced. One of these effects is the nearby presence of many other identical gravitational dipoles. In this case the many attractive and repulsive forces between the individual gravitational charges can create stable dynamical configurations. The initial study [18] suggests that such stable dynamical configurations of gravitational dipoles can exist in the tensionless, high energy limit [30], of relativistic string and membrane theory. The idea that strings should be regarded as composite systems of more fundamental point-like objects was first introduced by Thorn [31] in 1991. An oscillating gravitational dipole would give rise to time-dependent gravitoelectric and gravitomagnetic fields and, consequently, to gravitational waves in empty space.

Gravitomagnetism is a natural prediction of general relativity when gravitational currents are taken into account. However, the history of gravitomagnetism begins before the advent of general relativity. Maxwell himself [19], and some time later Heaviside [20], inspired by the analogy between Newtonian gravity and electrostatics, dedicated part of their work in searching an evidence of the existence of gravitomagnetism. Formal recent developments of gravitomagnetism in the framework of general relativity [21] and in the non-relativistic Newtonian framework [22] can be found in the literature. In particular, it was verified in [21] that in the weak field approximation the gravitomagnetic equation in empty space derived using general relativity is exactly analogous to Faraday's law of electromagnetic induction in electrodynamics. In this work we present an initial study of the classical Hamiltonian formalism for the action describing a massless relativistic particle moving in a background gravitodynamic potential. Our contributions to the gravitodynamic theory of this model are: a) we reveal that this model defines a genuine generally covariant system b) we reveal that this model defines a conformal theory c) we expose the fact that this gravitodynamic theory, formulated in a flat Euclidean space-time or in a noncommutative Snyder-like space-time, have non-vanishing Poisson brackets between the phase space variables describing the motion of the massless particle and the components of the vector potential that describes the background gravitodynamic field, which is an indication of the physical nature of the potential in the classical theory d) we give a formal derivation of the non-relativistic equation of motion for a massless particle moving in a background Newtonian gravitodynamic field. These subjects will be considered in section four.

As an introduction to the gravitodynamic theory, the paper also contains the following developments. In section two we present a new scale invariance of the Hamiltonian that describes a free massless relativistic particle. We then show how this invariance can be used to perform a transition to a phase space with Snyder-like brackets. This result acquires importance with connection to the fact that the massless particle is a prototype of general relativity and also of string and membrane theories. Section two closes with an explicit verification that, modulo the Hamiltonian constraint, the Snyder brackets we derived preserve the usual conformal invariance of the free massless particle action. One immediate consequence of this verification is that in the classical free massless particle theory we can use the de Sitter/Snyder-Yang connection in place of the AdS/CFT connection, as proposed in [7], with no information loss because conformal invariance in the Snyder-like classical space-time is restored when the Hamiltonian constraint is imposed.

As has been demonstrated [11,12], the free massless particle, the harmonic oscillator and the Hydrogen atom are some of the many dual systems that have a unified description given by the two-time physics model. For this reason, it is theoretically important to investigate the existence of the same type of Snyder-like brackets in the two-time physics formalism. This task is taken in section three, where it is shown that the same type of scale invariance we found for the free massless particle Hamiltonian can be found for the two-time physics Hamiltonian. The usual canonical Poisson brackets and two different types of Snyder-like brackets are then derived using this invariance. Section three concludes with an explicit verification that one of the two Snyder-like sets of brackets we derived preserve the Lorentz invariance of the 2T model. The implications of the other Snyder-like set, which seems to be related with the relativistic physics of mesoscopic systems, remains to be investigated.

Section four contains our contributions to the Newtonian gravitodynamic theory. Gravitomagnetism is turning into a subject of appreciable interest as a consequence of the rapidly increasing technological refinement in the measuring and detecting devices used in quantum gravity experiments. It has been pointed out [21] that gravitomagnetism can explain the anomalous acceleration observed on the Pioneer spacecraft and also contribute [22] to the experimentally observed Lense-Thirring effect (dragging of the inertial frame in a gravitational field). It is then interesting to investigate these and other effects in the context of a relativistic Newtonian gravitodynamic theory and section four presents some first steps in this direction. Concluding remarks appear in section five.

## 2 Relativistic Particles

A relativistic particle describes in space-time a one-parameter trajectory  $x^\mu(\tau)$ . A possible form of the action is the one proportional to the arc length traveled by the particle and given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2} \quad (2.1)$$

$\tau$  is an arbitrary parameter along the particle's world-line,  $m$  is the particle's mass and  $ds^2 = -\delta_{\mu\nu}dx^\mu dx^\nu$ . We work in a  $D$ -dimensional Euclidean space-time with  $\mu = 1, \dots, D$ . A dot denotes derivatives with respect to  $\tau$  and we use units in which  $\hbar = c = 1$ .

Action (2.1) is invariant under the Poincaré transformation

$$\delta x^\mu = a^\mu + \omega_\nu^\mu x^\nu \quad (2.2)$$

where  $a^\mu$  is a constant vector and  $\omega_{\mu\nu} = -\omega_{\nu\mu}$  is a constant matrix. As a consequence of the invariance of action (2.1) under transformation (2.2), the following field can be defined in space-time

$$V = a^\mu p_\mu - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} \quad (2.3)$$

where  $p_\mu$  is the particle's momentum and  $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ . Introducing the fundamental Poisson brackets

$$\{p_\mu, p_\nu\} = 0 \quad (2.4a)$$

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu} \quad (2.4b)$$

$$\{x_\mu, x_\nu\} = 0 \quad (2.4c)$$

we find that the generators of the field  $V$  obey the algebra

$$\{p_\mu, p_\nu\} = 0 \quad (2.5a)$$

$$\{p_\mu, M_{\nu\lambda}\} = \delta_{\mu\nu} p_\lambda - \delta_{\mu\lambda} p_\nu \quad (2.5b)$$

$$\{M_{\mu\nu}, M_{\rho\lambda}\} = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \quad (2.5c)$$

This is the Poincaré space-time algebra in  $D$  dimensions. Action (2.1) is also invariant under the reparametrizations of the world-line

$$\tau \rightarrow \tau' = f(\tau) \quad (2.6)$$

where  $f$  is an arbitrary continuous function of  $\tau$ . As a consequence of its invariance under transformation (2.6), the particle action (2.1) defines the simplest possible generally covariant physical system.

Action (2.1) is obviously inadequate to study the massless limit of relativistic particle theory and so we must find an alternative action. Such an action can be easily computed by treating the relativistic particle as a constrained system. In the transition to the Hamiltonian formalism, action (2.1) gives the canonical momentum

$$p_\mu = \frac{m}{\sqrt{-\dot{x}^2}} \dot{x}_\mu \quad (2.7)$$

and this momentum gives rise to the primary constraint

$$\phi = \frac{1}{2}(p^2 + m^2) = 0 \quad (2.8)$$

In this work we follow Dirac's [15] convention that a constraint is set equal to zero only after all calculations have been performed. The canonical Hamiltonian corresponding to action (2.1),  $H = p.\dot{x} - L$ , identically vanishes. Dirac's Hamiltonian for the relativistic particle is then

$$H_D = H + \lambda\phi = \frac{1}{2}\lambda(p^2 + m^2) \quad (2.9)$$

where  $\lambda(\tau)$  is a Lagrange multiplier, to be interpreted as an independent variable. We see from (2.8) and (2.9) that the dynamics of the relativistic particle is not governed by a true Hamiltonian but rather by a Hamiltonian constraint. The Lagrangian that corresponds to (2.9) is

$$L = p.\dot{x} - \frac{1}{2}\lambda(p^2 + m^2) \quad (2.10)$$

Solving the equation of motion for  $p_\mu$  that follows from (2.10) and inserting the result back in it, we obtain the particle action

$$S = \int d\tau \left( \frac{1}{2}\lambda^{-1}\dot{x}^2 - \frac{1}{2}\lambda m^2 \right) \quad (2.11)$$

In action (2.11),  $\lambda(\tau)$  can be associated [23] to a "world-line metric"  $\gamma_{\tau\tau}$ ,  $\lambda(\tau) = [-\gamma_{\tau\tau}(\tau)]^{\frac{1}{2}}$ , such that  $ds^2 = \gamma_{\tau\tau}d\tau d\tau$ . In (2.11),  $\lambda(\tau)$  is an "einbein" field. In more dimensions, the "vielbein"  $e_\mu^a$  is an alternative description of the metric tensor. In this context, the particle mass  $m$  plays the role of a (0+1)-dimensional "cosmological constant". Action (2.11) is classically equivalent to action (2.1). This can be checked in the following way. If we solve the classical equation of motion for  $\lambda(\tau)$  that follows from (2.11) we get the result  $\lambda = \pm(\sqrt{-\dot{x}^2}/m)$ . Inserting the solution with the positive sign in (2.11), it becomes identical to (2.1). The great advantage of action (2.11) is that it has a smooth transition to the  $m = 0$  limit.

The general covariance of action (2.11) manifests itself through invariance under the transformation

$$\delta x^\mu = \epsilon \dot{x}^\mu \quad (2.12a)$$

$$\delta \lambda = \frac{d}{d\tau}(\epsilon \lambda) \quad (2.12b)$$

where  $\epsilon(\tau)$  is an arbitrary infinitesimal parameter. Varying  $x^\mu$  in (2.11) we obtain the classical equation for free motion

$$\frac{d}{d\tau} \left( \frac{\dot{x}_\mu}{\lambda} \right) = \dot{p}_\mu = 0 \quad (2.13)$$

Now we make a transition to the massless limit. This limit is described by the action

$$S = \frac{1}{2} \int d\tau \lambda^{-1} \dot{x}^2 \quad (2.14)$$

Action (2.14) is invariant under the Poincaré transformation (2.2) with  $\delta\lambda = 0$  and under the infinitesimal reparametrization (2.12). However, action (2.14) exhibits other invariances which are not shared by the massive particle action (2.11). These are the invariance under global scale transformations

$$\delta x^\mu = \alpha x^\mu \quad (2.15a)$$

$$\delta\lambda = 2\alpha\lambda \quad (2.15b)$$

where  $\alpha$  is a constant, and invariance under the conformal transformations

$$\delta x^\mu = (2x^\mu x^\nu - \delta^{\mu\nu} x^2) b_\nu \quad (2.16a)$$

$$\delta\lambda = 4\lambda x \cdot b \quad (2.16b)$$

where  $b_\mu$  is a constant vector. As a consequence of the presence of these two additional invariances, in the massless particle case the field (2.3) can be extended to

$$V = a^\mu p_\mu - \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} + \alpha D + b^\mu K_\mu \quad (2.17)$$

with the additional generators

$$D = x^\mu p_\mu \quad (2.18)$$

$$K_\mu = (2x_\mu x^\nu - \delta_\mu^\nu x^2) p_\nu \quad (2.19)$$

which correspond to invariances (2.15) and (2.16), respectively. The generators of the vector field (2.17) now define the space-time algebra

$$\{p_\mu, p_\nu\} = 0 \quad (2.20a)$$

$$\{p_\mu, M_{\nu\lambda}\} = \delta_{\mu\nu} p_\lambda - \delta_{\mu\lambda} p_\nu \quad (2.20b)$$

$$\{M_{\mu\nu}, M_{\lambda\rho}\} = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \quad (2.20c)$$

$$\{D, D\} = 0 \quad (2.20d)$$

$$\{D, p_\mu\} = p_\mu \quad (2.20e)$$

$$\{D, M_{\mu\nu}\} = 0 \quad (2.20f)$$

$$\{D, K_\mu\} = -K_\mu \quad (2.20g)$$

$$\{p_\mu, K_\nu\} = -2\delta_{\mu\nu}D + 2M_{\mu\nu} \quad (2.20h)$$

$$\{M_{\mu\nu}, K_\lambda\} = \delta_{\nu\lambda}K_\mu - \delta_{\lambda\mu}K_\nu \quad (2.20i)$$

$$\{K_\mu, K_\nu\} = 0 \quad (2.20j)$$

The algebra (2.20) is the conformal space-time algebra [29]. The free massless particle theory defined by action (2.14) is a conformal theory in  $D$  space-time dimensions.

The classical equation of motion for  $x^\mu$  that follows from action (2.14) is identical to (2.13). The equation of motion for  $\lambda$  gives the condition  $\dot{x}^2 = 0$ , which tells us that a free massless relativistic particle moves at the speed of light. As a consequence of this, it becomes impossible to solve for  $\lambda(\tau)$  from its equation of motion. In the massless theory the value of  $\lambda(\tau)$  is completely arbitrary.

In the transition to the Hamiltonian formalism the massless action (2.14) gives the canonical momenta

$$p_\lambda = 0 \quad (2.21)$$

$$p_\mu = \frac{\dot{x}_\mu}{\lambda} \quad (2.22)$$

and the canonical Hamiltonian

$$H = \frac{1}{2}\lambda p^2 \quad (2.23)$$

Equation (2.21) is a primary constraint. Introducing the Lagrange multiplier  $\xi(\tau)$  for this constraint we can write the Dirac Hamiltonian

$$H_D = \frac{1}{2}\lambda p^2 + \xi p_\lambda \quad (2.24)$$

Requiring the dynamical stability of constraint (2.21),  $\dot{p}_\lambda = \{p_\lambda, H_D\} = 0$ , we obtain the secondary constraint

$$\phi = \frac{1}{2}p^2 = 0 \quad (2.25)$$

Constraint (2.25) has a vanishing Poisson bracket with itself and with constraint (2.21), being therefore a first-class constraint. As a consequence, among the  $D$  variables  $x^\mu$  that appear in action (2.14) only  $D - 1$  correspond to real physical degrees of freedom. Constraint (2.21) generates translations in the arbitrary variable  $\lambda(\tau)$  and can therefore be dropped from the formalism.

Now we present a new invariance of the Hamiltonian formalism. The massless particle Hamiltonian (2.23) is invariant under the transformations

$$p_\mu \rightarrow \tilde{p}_\mu = \exp\{-\beta(\dot{x}^2)\}p_\mu \quad (2.26a)$$

$$\lambda \rightarrow \exp\{2\beta(\dot{x}^2)\}\lambda \quad (2.26b)$$

where  $\beta$  is an arbitrary function of  $\dot{x}^2$ . From the equation (2.22) for the canonical momentum we find that  $x^\mu$  should transform as

$$x^\mu \rightarrow \tilde{x}^\mu = \exp\{\beta(\dot{x}^2)\}x^\mu \quad (2.27)$$

when  $p_\mu$  transforms as in (2.26a).

Consider now the bracket structure that transformations (2.26a) and (2.27) induce in the massless particle's phase space. Taking  $\beta(\dot{x}^2) = \beta(\lambda^2 p^2)$  in transformations (2.26a) and (2.27), and retaining only the linear terms in  $\beta$  in the exponentials, we find that the new transformed canonical variables  $(\tilde{x}_\mu, \tilde{p}_\mu)$  obey the brackets

$$\{\tilde{p}_\mu, \tilde{p}_\nu\} = 0 \quad (2.28a)$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = (1 + \beta)[\delta_{\mu\nu}(1 - \beta) - \{x_\mu, \beta\}p_\nu] \quad (2.28b)$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = (1 + \beta)[x_\mu\{\beta, x_\nu\} - x_\nu\{\beta, x_\mu\}] \quad (2.28c)$$

written in terms of the old canonical variables. These general brackets obey the non trivial Jacobi identities  $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{x}_\lambda) = 0$  and  $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{p}_\lambda) = 0$ , where

$$(a, b, c) = \{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\}$$

As we will see in the following, with a suitable choice for  $\beta$ , the brackets (2.28) lead to extensions of the fundamental Poisson brackets that can be used to define in a noncommutative Snyder space-time in the quantum theory..

It turns out that the non-commutative space-time geometry induced by the brackets (2.28) will be completely determined by the choice  $\beta(\dot{x}^2) = \beta(\lambda^2 p^2) = \frac{1}{2}\lambda^2 p^2$ . This is because classical Poisson brackets, as quantum commutators, satisfy the property  $\{A^n, B\} = nA^{n-1}\{A, B\}$ . If we consider, for instance,  $\beta = (\lambda^2 p^2)^2$  and compute  $\{\beta, x^\mu\}$  we will find  $\{\beta, x^\mu\} = 2\lambda^2 p^2 \{\lambda^2 p^2, x^\mu\}$ , and the right hand side vanishes when constraint (2.25) is imposed. Similarly, all higher order terms will vanish when (2.25) is imposed, and the space-time geometry is completely determined by the case  $\beta = \frac{1}{2}\lambda^2 p^2$ . Computing the brackets (2.28b) and (2.28c) for this form of  $\beta$ , and finally imposing constraint (2.25), we arrive at the brackets

$$\{\tilde{p}_\mu, \tilde{p}_\nu\} = 0 \quad (2.29a)$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = \delta_{\mu\nu} - \lambda^2 p_\mu p_\nu \quad (2.29b)$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = -\lambda^2(x_\mu p_\nu - x_\nu p_\mu) \quad (2.29c)$$

while, again imposing constraint (2.25), the transformation equations (2.26a) and (2.27) become the identity transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu \quad (2.30a)$$

$$p_\mu \rightarrow \tilde{p}_\mu = p_\mu \quad (2.30b)$$

Using equations (2.30) in (2.29), we can then write down the brackets

$$\{p_\mu, p_\nu\} = 0 \quad (2.31a)$$

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu} - \lambda^2 p_\mu p_\nu \quad (2.31b)$$

$$\{x_\mu, x_\nu\} = -\lambda^2(x_\mu p_\nu - x_\nu p_\mu) \quad (2.31c)$$

In the transition to the quantum theory the brackets (2.31) will exactly reproduce the Snyder commutators [5] proposed in 1947 (with  $\lambda^2$  playing the role of the noncommutativity parameter  $\theta$ ).

The brackets (2.31) obey all Jacobi identities among the canonical variables. Since in the massless theory the value of  $\lambda(\tau)$  is arbitrary, as we saw above, we can use the reparametrization invariance (2.12) to choose a gauge in which  $\lambda = 1$ . We then end up with the brackets

$$\{p_\mu, p_\nu\} = 0 \quad (2.32a)$$

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu} - p_\mu p_\nu \quad (2.32b)$$

$$\{x_\mu, x_\nu\} = -(x_\mu p_\nu - x_\nu p_\mu) \quad (2.32c)$$

Computing the conformal algebra using the brackets (2.32) instead of the Poisson brackets (2.4) we obtain

$$\{p_\mu, p_\nu\} = 0 \quad (2.33a)$$

$$\{p_\mu, M_{\nu\lambda}\} = \delta_{\mu\nu} p_\lambda - \delta_{\mu\lambda} p_\nu \quad (2.33b)$$

$$\{M_{\mu\nu}, M_{\lambda\rho}\} = \delta_{\nu\lambda} M_{\mu\rho} + \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\lambda} M_{\nu\rho} \quad (2.33c)$$

$$\{D, D\} = 0 \quad (2.33d)$$

$$\{D, p_\mu\} = p_\mu - p_\mu p^2 \quad (2.33e)$$

$$\{D, M_{\mu\nu}\} = 0 \quad (2.33f)$$

$$\{D, K_\mu\} = -K_\mu + 2K_\mu p^2 \quad (2.33g)$$

$$\{p_\mu, K_\nu\} = -2\delta_{\mu\nu}D + 2M_{\mu\nu} + 2p_\mu x_\nu p^2 \quad (2.33h)$$

$$\{M_{\mu\nu}, K_\lambda\} = \delta_{\nu\lambda}K_\mu - \delta_{\lambda\mu}K_\nu \quad (2.33i)$$

$$\{K_\mu, K_\nu\} = 2x^2 M_{\mu\nu} p^2 \quad (2.33j)$$

Imposing again constraint (2.25) we see that the brackets (2.32) preserve the structure (2.20) of the  $D$  dimensional conformal space-time algebra. The brackets (2.32) then have a peaceful coexistence with conformal invariance in free massless particle theory. In the next section we will verify that the brackets (2.32) also preserve the  $(D+2)$  dimensional Lorentz invariance of the two-time physics model.

### 3 Two-time Physics

As we mentioned in the introduction, the free massless particle is one of the physical systems that have a unified description given by the two-time physics model. The massless particle Hamiltonian in  $D$  dimensions is obtained by making two gauge choices in the  $D+2$  dimensional Hamiltonian formalism for the 2T model. It is then interesting to investigate the existence, in 2T physics, of the  $D+2$  extension of the  $D$  dimensional brackets (2.32), an extension which is still lacking in the literature. This is done in this section.

The central idea in two-time physics [9,10,11,12] is to introduce a new gauge invariance in phase space by gauging the duality of the quantum commutator  $[X_M, P_N] = i\delta_{MN}$ . This procedure leads to the symplectic  $\text{Sp}(2, \mathbb{R})$  gauge invariance. To remove the distinction between position and momenta we set  $X_1^M = X^M$  and  $X_2^M = P^M$  and define the doublet  $X_i^M = (X_1^M, X_2^M)$ . The local  $\text{Sp}(2, \mathbb{R})$  acts as

$$\delta X_i^M(\tau) = \epsilon_{ik} \omega^{kl}(\tau) X_l^M(\tau) \quad (3.1)$$

$\omega^{ij}(\tau)$  is a symmetric matrix containing three local parameters and  $\epsilon_{ij}$  is the Levi-Civita symbol that serves to raise or lower indices. The  $\text{Sp}(2, \mathbb{R})$  gauge field  $A^{ij}$  is symmetric in  $(i, j)$  and transforms as

$$\delta A^{ij} = \partial_\tau \omega^{ij} + \omega^{ik} \epsilon_{kl} A^{lj} + \omega^{jk} \epsilon_{kl} A^{il} \quad (3.2)$$

The covariant derivative is

$$D_\tau X_i^M = \partial_\tau X_i^M - \epsilon_{ik} A^{kl} X_l^M \quad (3.3)$$

An action invariant under the  $\text{Sp}(2, \mathbb{R})$  gauge symmetry is

$$S = \frac{1}{2} \int d\tau (D_\tau X_i^M) \epsilon^{ij} X_j^N \eta_{MN}$$

After an integration by parts this action can be written as

$$\begin{aligned} &= \int d\tau (\partial_\tau X_1^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N) \eta_{MN} \\ &= \int d\tau [\dot{X} \cdot P - (\frac{1}{2} \lambda_1 P^2 + \lambda_2 P \cdot X + \frac{1}{2} \lambda_3 X^2)] \end{aligned} \quad (3.4)$$

where  $A^{11} = \lambda_3$ ,  $A^{12} = A^{21} = \lambda_2$ ,  $A^{22} = \lambda_1$  and the canonical Hamiltonian is

$$H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 P \cdot X + \frac{1}{2} \lambda_3 X^2 \quad (3.5)$$

The equations of motion for the  $\lambda$ 's give the primary constraints

$$\phi_1 = \frac{1}{2} P^2 = 0 \quad (3.6)$$

$$\phi_2 = P \cdot X = 0 \quad (3.7)$$

$$\phi_3 = \frac{1}{2} X^2 = 0 \quad (3.8)$$

and therefore we can not solve for the  $\lambda$ 's from their equations of motion. The values of the  $\lambda$ 's in action (3.4) are completely arbitrary. If we consider the Euclidean, or the Minkowski metric, as the background space, we find that the surface defined by the constraint equations (3.6)-(3.8) is trivial. The only metric giving a non-trivial surface and avoiding the ghost problem is the flat metric with two time coordinates. We then introduce an extra time-like dimension and an extra space-like dimension and work in a  $(D+2)$  dimensional space-time.

We use the fundamental Poisson brackets

$$\{P_M, P_N\} = 0 \quad (3.9a)$$

$$\{X_M, P_N\} = \delta_{MN} \quad (3.9b)$$

$$\{X_M, X_N\} = 0 \quad (3.9c)$$

where  $M, N = 1, \dots, D+2$ . and verify that constraints (3.6)-(3.8) obey the algebra

$$\{\phi_1, \phi_2\} = -2\phi_1 \quad (3.10a)$$

$$\{\phi_1, \phi_3\} = -\phi_2 \quad (3.10b)$$

$$\{\phi_2, \phi_3\} = -2\phi_3 \quad (3.10c)$$

These equations show that all constraints  $\phi$  are first class. Equations (3.10) represent the Lie algebra of the symplectic  $\text{Sp}(2, \mathbb{R})$  group, which is the gauge group of the two-time physics model. The gauge algebra (3.10) has an analogous [8] algebra which is the gauge algebra of the  $\text{SL}(2, \mathbb{R})$  model [26]. The  $\text{SL}(2, \mathbb{R})$  model mimics the gauge symmetry of general relativity and has the interesting feature that the dynamics is not governed by a true Hamiltonian but rather by two Hamiltonian constraints. If we redefine the constraints as  $H_1 = \phi_1$ ,  $H_2 = -\phi_3$  and  $D = \phi_2$  the algebra (3.10) becomes

$$\{H_1, D\} = -2H_1 \quad (3.11a)$$

$$\{H_1, H_2\} = D \quad (3.11b)$$

$$\{H_2, D\} = -2H_2 \quad (3.11c)$$

which is the gauge algebra of the  $\text{SL}(2, \mathbb{R})$  model.

Action (3.4) is in first order form. We can go to a second order formalism by eliminating the canonical momenta. The classical equation of motion for  $P_M$  that follows from action (3.4) has the solution

$$P_M = \frac{1}{\lambda_1}(\dot{X}_M - \lambda_2 X_M) \quad (3.12)$$

Inserting this solution in action (3.4), we obtain the alternative 2T action

$$S = \int d\tau \left[ \frac{1}{2\lambda_1}(\dot{X} - \lambda_2 X)^2 - \frac{1}{2}\lambda_3 X^2 \right] \quad (3.13)$$

This form of the action is a generalization of (0+1) dimensional gravity to (0+1) dimensional conformal gravity [13,14]. An action analogous to (3.13) was independently derived in [29]. In the canonical formalism for action (3.13) the constraints (3.6)-(3.8) appear as secondary constraints that are necessary for the dynamical stability of the primary constraints  $p_{\lambda_1} = p_{\lambda_2} = p_{\lambda_3} = 0$ . Because constraints (3.6)-(3.8) are first-class, only  $D - 1$  of the  $D + 2$  variables  $X_M$  that appear in the 2T action (3.13) correspond to real physical degrees of freedom. The conformal gravity action (3.13) and the free massless particle action (2.14) therefore describe the same number of physical degrees of freedom.

Now let us see how the  $D + 2$  version of the free massless particle brackets (2.32) can be found in two-time physics. It can be verified that the Hamiltonian (3.5) is invariant under the transformations

$$X^M \rightarrow \tilde{X}^M = \exp\{\beta(\dot{X}^2)\}X^M \quad (3.14a)$$

$$P_M \rightarrow \tilde{P}_M = \exp\{-\beta(\dot{X}^2)\}P_M \quad (3.14b)$$

$$\lambda_1 \rightarrow \exp\{2\beta(\dot{X}^2)\}\lambda_1 \quad (3.14c)$$

$$\lambda_2 \rightarrow \lambda_2 \quad (3.14d)$$

$$\lambda_3 \rightarrow \exp\{-2\beta(\dot{X}^2)\}\lambda_3 \quad (3.14e)$$

$\beta(\dot{X}^2)$  is an arbitrary function of  $\dot{X}^2$ . The transformation equations (3.14) are the 2T extensions of the invariance (2.26), (2.27) we found for the free massless particle. The existence of this invariance of the 2T Hamiltonian opens the possibility of the existence of a  $D + 2$  dimensional extension of the free massless particle brackets (2.32). But in the 2T model we have an enlarged gauge invariance and therefore new possibilities.

From the equation (3.12) for the canonical momentum we find

$$\dot{X}^2 = \lambda_1^2 P^2 + 2\lambda_1 \lambda_2 P \cdot X + \lambda_2^2 X^2 \quad (3.15)$$

and in the 2T Hamiltonian formalism the arbitrary function  $\beta(\dot{X}^2)$  that appears in transformation (3.14) must be a canonical field  $\beta(X, P)$ . Keeping only the linear terms in  $\beta$  in the transformation (3.14), after some algebra we arrive at the brackets

$$\{\tilde{P}_M, \tilde{P}_N\} = (\beta - 1)[\{P_M, \beta\}P_N + \{\beta, P_N\}P_M] + \{\beta, \beta\}P_M P_N \quad (3.16a)$$

$$\begin{aligned} \{\tilde{X}_M, \tilde{P}_N\} &= (1 + \beta)[\delta_{MN}(1 - \beta) - \{X_M, \beta\}P_N] \\ &+ (1 - \beta)X_M\{\beta, P_N\} - X_M X_N\{\beta, \beta\} \end{aligned} \quad (3.16b)$$

$$\{\tilde{X}_M, \tilde{X}_N\} = (1 + \beta)[X_M\{\beta, X_N\} - X_N\{\beta, X_M\}] + X_M X_N\{\beta, \beta\} \quad (3.16c)$$

There are three important choices for  $\beta$  and each one leads to a different set of brackets. Consider first the simplest one, which corresponds to imposing constraints (3.6)-(3.8) in expression (3.15). We have  $\dot{X}^2 = 0$  and we can then choose  $\beta(\dot{X}^2) = 0$  in expressions (3.16). We then find the brackets

$$\{P_M, P_N\} = 0 \quad (3.17a)$$

$$\{X_M, P_N\} = \delta_{MN} \quad (3.17b)$$

$$\{X_M, X_N\} = 0 \quad (3.17c)$$

where we used the fact that transformations (3.14) become the identity transformation when  $\beta(\dot{X}^2) = 0$ . Brackets (3.17) are identical to the usual Poisson brackets (3.9).

To obtain the extension of the massless particle commutators (2.32) in the 2T model we impose constraints (3.7) and (3.8) in equation (3.15). We can then choose the form  $\beta(\dot{X}^2) = \frac{1}{2}\lambda_1^2 P^2$ . Inserting this form for  $\beta$  in the general brackets (3.16), imposing constraint (3.6) at the end of the calculation so that transformations (3.14) become the identity transformation, and choosing the gauge  $\lambda_1 = 1$ , we arrive at the brackets

$$\{P_M, P_N\} = 0 \quad (3.18a)$$

$$\{X_M, P_N\} = \delta_{MN} - P_M P_N \quad (3.18b)$$

$$\{X_M, X_N\} = -(X_M P_N - X_N P_M) \quad (3.18c)$$

which are the  $D + 2$  extensions of the free massless particle brackets (2.32).

Another interesting choice is to impose constraints (3.6) and (3.7) first in expression (3.15). A convenient form for  $\beta$  is now  $\beta(\dot{X}^2) = \frac{1}{2}\lambda_2^2 X^2$ . Inserting this form for  $\beta$  in the brackets (3.16), imposing constraint (3.8) at the end of the calculation, and choosing the gauge  $\lambda_2 = 1$ , we get the brackets

$$\{P_M, P_N\} = (X_M P_N - X_N P_M) \quad (3.19a)$$

$$\{X_M, P_N\} = \delta_{MN} + X_M X_N \quad (3.19b)$$

$$\{X_M, X_N\} = 0 \quad (3.19c)$$

which are dual to the brackets (3.18).

The brackets (3.18) can be used to generate a non-linear realization of the classical  $\text{Sp}(2, \mathbb{R})$  gauge algebra (3.10). Computing the algebra of the first class constraints (3.6)-(3.8) using brackets (3.18) we get the new algebra

$$\{\phi_1, \phi_2\} = -2\phi_1 + 4\phi_1^2 \quad (3.20a)$$

$$\{\phi_1, \phi_3\} = -\phi_2 + 2\phi_1\phi_2 \quad (3.20b)$$

$$\{\phi_2, \phi_3\} = -2\phi_3 + \phi_2^2 \quad (3.20c)$$

and, by analogy, the gauge algebra (3.11) of the  $\text{SL}(2, \mathbb{R})$  model becomes

$$\{H_1, D\} = -2H_1 + 4H_1^2 \quad (3.21a)$$

$$\{H_1, H_2\} = D - 2H_1 D \quad (3.21b)$$

$$\{H_2, D\} = -2H_2 + D^2 \quad (3.21c)$$

The brackets (3.18) are therefore associated to an enlarged gauge algebra of the 2T model and of the  $SL(2, R)$  model.

At this point it is interesting to consider the  $(D + 2)$  dimensional Lorentz invariance of the 2T model. The generators of this symmetry are given by [11]

$$\begin{aligned} L_{MN} &= \epsilon^{ij} X_i^M X_j^N \\ &= X^M P^N - X^N P^M \end{aligned} \quad (3.22)$$

and satisfy the bracket

$$\{L_{MN}, L_{RS}\} = \eta_{MR} L_{NS} + \eta_{NS} L_{MR} - \eta_{NR} L_{MS} - \eta_{MS} L_{NR} \quad (3.23)$$

The generators (3.22) contain the full physical information of the theory and are gauge invariant [11]. They can therefore be computed in any gauge with identical results. Following [11] we then choose the basis  $X_M = (X_+, X_-, x_\mu)$  with the metric  $\eta_{MN}$  taking the values  $\eta_{+-} = -1$  and  $\eta_{\mu\nu} = \delta_{\mu\nu}$ . We choose the two gauge conditions  $X_+ = 1$  and  $P_+ = 0$  and solve the two constraints  $P \cdot X = 0$  and  $X^2 = 0$  for  $X_-$  and  $P_-$ . The final expressions are [11]

$$M = (+, -, \mu) \quad (3.24a)$$

$$X_M = (1, \frac{1}{2}x^2, x_\mu) \quad (3.24b)$$

$$P_M = (0, p \cdot x, p_\mu) \quad (3.24c)$$

The Lorentz generators  $L_{MN}$  in this gauge are then given by

$$L_{++} = L_{--} = 0 \quad (3.25a)$$

$$L_{+-} = -L_{-+} = p \cdot x \quad (3.25b)$$

$$L_{+\mu} = -L_{\mu+} = p_\mu \quad (3.25c)$$

$$L_{-\mu} = -L_{\mu-} = \frac{1}{2}x^2 p_\mu - p \cdot x x_\mu \quad (3.25d)$$

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad (3.25e)$$

Using the  $D$  dimensional brackets (2.32) to compute the  $D + 2$  dimensional Lorentz algebra generated by these  $L_{MN}$  we find

$$\{L_{+\mu}, L_{+\nu}\} = 0 \quad (3.26a)$$

$$\{L_{+\nu}, L_{\nu\lambda}\} = \delta_{\mu\nu} L_{+\lambda} - \delta_{\mu\lambda} L_{+\nu} \quad (3.26b)$$

$$\{L_{\mu\nu}, L_{\nu\rho}\} = \delta_{\nu\lambda}L_{\mu\rho} + \delta_{\mu\rho}L_{\nu\lambda} - \delta_{\nu\rho}L_{\mu\lambda} - \delta_{\mu\lambda}L_{\nu\rho} \quad (3.26c)$$

$$\{L_{+-}, L_{+-}\} = 0 \quad (3.26d)$$

$$\{L_{+-}, L_{+\mu}\} = L_{+\mu} - L_{+\mu}p^2 \quad (3.26e)$$

$$\{L_{+-}, L_{\mu\nu}\} = 0 \quad (3.26f)$$

$$\{L_{+-}, L_{-\mu}\} = -L_{-\mu} + 2L_{-\mu}p^2 \quad (3.26g)$$

$$\{L_{+\mu}, L_{-\nu}\} = \delta_{\mu\nu}L_{+-} - L_{\mu\nu} - p_\mu x_\nu p^2 \quad (3.26h)$$

$$\{L_{\mu\nu}, L_{-\lambda}\} = \delta_{\nu\lambda}L_{-\mu} - \delta_{\lambda\mu}L_{-\nu} \quad (3.26i)$$

$$\{L_{-\mu}, L_{-\nu}\} = \frac{1}{2}x^2 L_{\mu\nu}p^2 \quad (3.26j)$$

where constraint  $\frac{1}{2}p^2 = 0$  must still be imposed. The  $D+2$  dimensional Lorentz algebra (3.26) of the two-time physics model becomes identical to the  $D$  dimensional conformal algebra (2.33) of the free massless particle if we make the identifications

$$L_{+\mu} = p_\mu \quad (3.27a)$$

$$L_{\mu\nu} = M_{\mu\nu} \quad (3.27b)$$

$$L_{+-} = D \quad (3.27c)$$

$$L_{-\mu} = -\frac{1}{2}K_\mu \quad (2.27d)$$

We then conclude that, again modulo the constraint, the Snyder brackets (2.32) also preserve the  $D+2$  dimensional Lorentz algebra of the two-time physics model, and that this algebra is identical to the  $D$  dimensional conformal algebra (2.33) of the free massless relativistic particle. This result is well known [12,29] to be true on the basis of the fundamental Poisson brackets (2.4). Now we have proved that it is also true on the basis of the Snyder brackets (2.32).

## 4 Newtonian Gravitodynamics

The Maxwell-Heaviside equations for Newtonian gravitodynamics were recently considered in [22]. They are given by

$$\vec{\nabla} \cdot \mathbf{E}_g = -4\pi G\rho \quad (4.1a)$$

$$\vec{\nabla} \times \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t} \quad (4.1b)$$

$$\vec{\nabla} \cdot \mathbf{B}_g = 0 \quad (4.1c)$$

$$\vec{\nabla} \times \mathbf{B}_g = -\frac{4\pi G}{c_g^2} \mathbf{j} + \frac{1}{c_g^2} \frac{\partial \mathbf{E}_g}{\partial t} \quad (4.1d)$$

In these equations  $G$  is Newton's constant,  $\rho$  is a mass density,  $\mathbf{j}$  is the vector density of mass currents,  $\mathbf{E}_g$  is the gravitoelectric vector field given by

$$\mathbf{E}_g = -\vec{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad (4.2)$$

and  $\vec{B}_g$  is the gravitomagnetic vector field

$$\mathbf{B}_g = \vec{\nabla} \times \mathbf{A} \quad (4.3)$$

$\Phi$  is the gravitational scalar potential and  $\mathbf{A}$  is the gravitational vector potential for this theory. For details and a brief review of the history of gravitodynamics see [22] and cited references.

Equations (4.1) contain the unknown constant  $c_g$ , which is the velocity of propagation of gravitational waves in empty space. Reasoning by analogy with electrodynamics, the author in [22] suggests that the force equation for a particle of mass  $m$  moving with velocity  $\mathbf{v}$  in a gravitodynamic field should be

$$\mathbf{F} = m\mathbf{E}_g + m\mathbf{v} \times \mathbf{B}_g \quad (4.4)$$

In this section we show how a formalism analogous to the one developed in the previous two sections can be constructed in the case of a massless relativistic particle moving in a background gravitodynamic field. We also derive the massless limit of the non-relativistic equation (4.4).

In the same way as Maxwell's electrodynamics [24], the gravitodynamic field can be described with the help of a relativistic vector potential  $A_\mu = (\mathbf{A}, i\Phi)$ . We can then introduce the  $D$  dimensional action

$$S = \int d\tau \left( \frac{1}{2} \lambda^{-1} \dot{x}^2 + A \cdot \dot{x} \right) \quad (4.5)$$

for a massless relativistic particle moving in an external gravitodynamic field  $A_\mu = A_\mu(\tau)$ . Action (4.5) is invariant under the infinitesimal reparametrization

$$\delta x^\mu = \epsilon \dot{x}^\mu \quad (4.6a)$$

$$\delta\lambda = \frac{d}{d\tau}(\epsilon\lambda) \quad (4.6b)$$

$$\delta A_\mu = \epsilon \dot{A}_\mu \quad (4.6c)$$

where  $\epsilon(\tau)$  is an arbitrary parameter, because the Lagrangian in action (4.5) transforms as a total derivative,  $\delta L = \frac{d}{d\tau}(\epsilon L)$ , under transformations (4.6). Action (4.5) therefore describes a genuine generally covariant physical system.

Action (4.5) is also invariant under the Poincaré transformation

$$\delta x^\mu = a^\mu + \omega_\nu^\mu x^\nu \quad (4.7a)$$

$$\delta A^\mu = b^\mu + \omega_\nu^\mu A^\nu \quad (4.7b)$$

$$\delta\lambda = 0 \quad (4.7c)$$

under the scale transformation

$$\delta x^\mu = \alpha x^\mu \quad (4.8a)$$

$$\delta A^\mu = -\alpha A^\mu \quad (4.8b)$$

$$\delta\lambda = 2\alpha\lambda \quad (4.8c)$$

and under the conformal transformation

$$\delta x^\mu = (2x^\mu x^\nu - \delta^{\mu\nu} x^2) b_\nu \quad (4.9a)$$

$$\delta A^\mu = 2(x^\mu A^\nu - A^\mu x^\nu - A.x\delta^{\mu\nu}) b_\nu \quad (4.9b)$$

$$\delta\lambda = 4\lambda x.b \quad (4.9c)$$

The generally covariant gravitodynamic action (4.5) therefore defines a  $D$  dimensional conformal theory.

The physically relevant canonical momenta that follow from action (4.5) are

$$p_\lambda = 0 \quad (4.10)$$

$$p_\mu = \frac{\dot{x}_\mu}{\lambda} + A_\mu \quad (4.11)$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}\lambda(p_\mu - A_\mu)^2 \quad (4.12)$$

Equation (4.10) is a primary constraint. Introducing the Lagrange multiplier  $\chi(\tau)$  for this constraint, constructing the Dirac Hamiltonian  $H_D = H + \chi p_\lambda$  and requiring the dynamical stability of (4.10), we obtain the secondary constraint

$$\phi = \frac{1}{2}(p_\mu - A_\mu)^2 = 0 \quad (4.13)$$

Now, the gravitodynamic Hamiltonian (4.12) is invariant under the transformations

$$p_\mu \rightarrow \tilde{p}_\mu = \exp\{-\beta(\dot{x}^2)\}p_\mu \quad (4.14a)$$

$$\lambda \rightarrow \exp\{2\beta(\dot{x}^2)\}\lambda \quad (4.14b)$$

$$A_\mu \rightarrow \tilde{A}_\mu = \exp\{-\beta(\dot{x}^2)\}A_\mu \quad (4.14c)$$

and from expression (4.11) we find that  $x^\mu$  should transform as

$$x^\mu \rightarrow \tilde{x}^\mu = \exp\{\beta(\dot{x}^2)\}x^\mu \quad (4.14d)$$

when  $p_\mu$  transforms as in (4.14a). From (4.11) we also have

$$\dot{x}^2 = \lambda^2(p^2 - 2p \cdot A + A^2) \quad (4.15)$$

To proceed in a consistent way it is necessary to introduce, in addition to the brackets (2.4), the new brackets

$$\{A_\mu, x_\nu\} = A_\mu x_\nu \quad (4.16a)$$

$$\{A_\mu, p_\nu\} = -p_\nu A_\mu \quad (4.16b)$$

$$\{A_\mu, A_\nu\} = 0 \quad (4.16c)$$

The introduction of bracket (4.16b) will be justified later when we derive the massless limit of (4.4). The introduction of bracket (4.16a) is justified by the consistency of the formalism developed below. Brackets (4.16a) and (4.16b) are indications of the physical nature of the gravitodynamic potential  $A_\mu$  in classical physics.

Transformations (4.14) induce the general bracket structure in the transformed phase space

$$\{\tilde{p}_\mu, \tilde{p}_\nu\} = (\beta - 1)[\{p_\mu, \beta\}p_\nu + \{\beta, p_\nu\}p_\mu] + \{\beta, \beta\}p_\mu p_\nu \quad (4.17a)$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = (1 + \beta)[\delta_{\mu\nu}(1 - \beta) - \{x_\mu, \beta\}p_\nu]$$

$$+ (1 - \beta)x_\mu\{\beta, p_\nu\} - x_\mu x_\nu\{\beta, \beta\} \quad (4.17b)$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = (1 + \beta)[x_\mu\{\beta, x_\nu\} - x_\nu\{\beta, x_\mu\}] + \{\beta, \beta\}x_\mu x_\nu \quad (4.17c)$$

$$\{\tilde{A}_\mu, \tilde{x}_\nu\} = (1 - \beta)\{A_\mu, \beta\}x_\nu - (1 + \beta)\{\beta, x_\nu\}A_\mu - \{\beta, \beta\}A_\mu x_\nu \quad (4.17d)$$

$$\{\tilde{A}_\mu, \tilde{p}_\nu\} = -(1 + 2\beta + 2\beta^2)p_\nu A_\mu - (1 - 2\beta)\{A_\mu, \beta\}p_\nu - A_\mu\{\beta, p_\nu\} \quad (4.17e)$$

$$\{\tilde{A}_\mu, \tilde{A}_\nu\} = 0 \quad (4.17f)$$

From (4.15), and using the reparametrization invariance (4.6) to impose the gauge  $\lambda^2 = \frac{1}{2}$ , we can choose the arbitrary function  $\beta(\dot{x}^2)$  that appears in transformations (4.14) to be

$$\beta(\dot{x}^2) = \frac{1}{2}p^2 - p \cdot A + \frac{1}{2}A^2 \quad (4.18)$$

Computing the brackets (4.17) for this form of  $\beta$  in terms of the fundamental brackets (2.4) and (4.16), and imposing constraint (4.13) at the end of the calculation, we arrive at the brackets

$$\{\tilde{p}_\mu, \tilde{p}_\nu\} = 0 \quad (4.19a)$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = \delta_{\mu\nu} - (p_\mu - A_\mu)p_\nu \quad (4.19b)$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = -[x_\mu(p_\nu - A_\nu) - x_\nu(p_\mu - A_\mu)] \quad (4.19c)$$

$$\{\tilde{A}_\mu, \tilde{x}_\nu\} = (p_\nu - A_\nu)A_\mu \quad (4.19d)$$

$$\{\tilde{A}_\mu, \tilde{p}_\nu\} = -p_\nu A_\mu \quad (4.19e)$$

$$\{\tilde{A}_\mu, \tilde{A}_\nu\} = 0 \quad (4.19f)$$

Imposing now constraint (4.13) in the transformations (4.14a), (4.14c) and (4.14d), these become the identity transformations

$$p_\mu \rightarrow \tilde{p}_\mu = p_\mu \quad (4.20a)$$

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu \quad (4.20b)$$

$$x_\mu \rightarrow \tilde{x}_\mu = x_\mu \quad (4.20c)$$

We then end with the brackets

$$\{p_\mu, p_\nu\} = 0 \quad (4.21a)$$

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu} - (p_\mu - A_\mu)p_\nu \quad (4.21b)$$

$$\{x_\mu, x_\nu\} = -[x_\mu(p_\nu - A_\nu) - x_\nu(p_\mu - A_\mu)] \quad (4.21c)$$

$$\{A_\mu, x_\nu\} = (p_\nu - A_\nu)A_\mu \quad (4.21d)$$

$$\{A_\mu, p_\nu\} = -p_\nu A_\mu \quad (4.21e)$$

$$\{A_\mu, A_\nu\} = 0 \quad (4.21f)$$

which extend the free massless particle brackets (2.32) to the case where the massless particle interacts with a background gravitodynamic potential  $A_\mu(\tau)$ . Notice that bracket (4.21c), which will determine the space-time geometry in the quantum theory, is consistent with the minimal coupling prescription  $p_\mu \rightarrow p_\mu - A_\mu$  applied to the free bracket (2.32c). The brackets (4.21d) and (4.21e) are again indications of the physical nature of the gravitodynamic potential in classical physics, but now in a Snyder-like space-time.

Let us now derive the massless limit of the non-relativistic equation (4.4). To do this we introduce an explicit dependence of the potential  $A_\mu(\tau)$  on the particle's coordinates and rewrite it as  $A_\mu(x^\nu(\tau))$ . In this case  $\{A_\mu, p_\nu\} = \frac{\partial A_\mu}{\partial x^\nu} = -p_\nu A_\mu$  and this explains the bracket (4.16b) above. With this explicit dependence we obtain from action (4.5) the equation of motion

$$\dot{p}_\mu = \frac{\partial A_\nu}{\partial x^\mu} \dot{x}^\nu - \dot{A}_\mu \quad (4.19)$$

The spatial part of this equation reads

$$\dot{\mathbf{p}} = \vec{\nabla}(\mathbf{A} \cdot \dot{\mathbf{r}} - \Phi \dot{t}) - \dot{\mathbf{A}} \quad (4.20)$$

We can now use the reparametrization invariance (4.6) to choose the gauge  $\tau = t$ . Equation (4.20) then becomes

$$\frac{d\mathbf{p}}{dt} = \vec{\nabla}(\mathbf{A} \cdot \mathbf{v}) - \vec{\nabla}\Phi - \frac{d\mathbf{A}}{dt} \quad (4.21)$$

Using the formula  $\vec{\nabla}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \vec{\nabla})\mathbf{b} + (\mathbf{b} \cdot \vec{\nabla})\mathbf{a} + \mathbf{a} \times (\vec{\nabla} \times \mathbf{b}) + \mathbf{b} \times (\vec{\nabla} \times \mathbf{a})$  and taking derivatives with respect to the coordinates keeping the velocities fixed, we find that

$$\vec{\nabla}(\mathbf{A} \cdot \mathbf{v}) = (\mathbf{v} \cdot \vec{\nabla})\mathbf{A} + \mathbf{v} \times (\vec{\nabla} \times \mathbf{A}) \quad (4.22)$$

Substituting this relation in (4.21), it becomes

$$\frac{d\mathbf{p}}{dt} = (\mathbf{v} \cdot \vec{\nabla})\mathbf{A} + \mathbf{v} \times (\vec{\nabla} \times \mathbf{A}) - \vec{\nabla}\Phi - \frac{d\mathbf{A}}{dt} \quad (4.23)$$

But from vector analysis we now that

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{A} \quad (4.24)$$

and therefore

$$-\frac{d\mathbf{A}}{dt} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{A} = -\frac{\partial \mathbf{A}}{\partial t} \quad (4.25)$$

Now substituting this in (4.23), we obtain

$$\frac{d\mathbf{p}}{dt} = -\vec{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\vec{\nabla} \times \mathbf{A}) \quad (4.26)$$

With the identifications (4.2) and (4.3), equation (4.26) can be rewritten as

$$\frac{d\mathbf{p}}{dt} = \mathbf{E}_g + \mathbf{v} \times \mathbf{B}_g \quad (4.27)$$

which extends the validity of equation (4.4) to the case of massless particles.

## 5 Concluding remarks

As is well known, in the canonical quantization procedure for a classical theory, one of the steps to the quantum theory is to define the fundamental commutators among the quantum operators that will describe the dynamics. The recipe is that these commutators are defined by multiplying by an “i” the values of the corresponding classical Poisson brackets. The results of this work then present an alternative possibility of constructing the quantum mechanics of the free massless relativistic particle, of the two-time physics model and of the Newtonian gravitodynamic theory. In these alternative formulations of the quantized theory the fundamental commutator relations are not of the Heisenberg type but are Snyder commutators.

In each of the above mentioned theories there are important reasons for investigating a quantum mechanics based on Snyder commutators. The free massless particle is only one of the many dual physical systems that have a unified description given by the two-time physics model. The list includes the harmonic oscillator, the Hydrogen atom, the particle moving in a de Sitter space, the particle moving in arbitrary attractive and repulsive potentials and possibly others still unknown. As for the free massless particle, Snyder brackets may also be hidden in the dynamics of all these dual physical systems, possibly in different space-time dimensions. As we saw in section three, the Snyder brackets for the two-time physics model enlarge the duality gauge algebra of the model and consequently may also enlarge the number of dual systems it can describe,

while still preserving its  $D + 2$  dimensional Lorentz invariance. As we saw in this work, the gravitodynamic theory brings with it a clear comprehension of the gravitational Aharonov-Bohm effect (also called the Aharonov-Carmi effect [28]). However, Snyder brackets for the gravitodynamic theory open the possibility of the existence of entirely new and unexpected gravitational effects, with no parallel in electrodynamics. Above all advances, relativistic quantum gravitodynamics in a noncommutative space-time can bring with it a clue of which is the fundamental physical object occupying the Planck area.

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